Charge-Current Filament Model in a Tokamak

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(Submitted 25 August 2003)

I view the plasma as a collection of filaments to which are attributed both a finite charge and a finite current. As in Taylor (1993), an equation for the current profile is determined by maximizing the entropy of this collection subject to fixed total energy. I examine in detail the case where the amount of charge each filament carries is infinitesimally small.

Plasma is created in a tokamak by a powerful electric field pointing in the toroidal direction. Because of energy transport, some regions are hotter than others. Temperature determines resistivity, which determines current. Currents introduce magnetic fields, which exert forces on the energy-carrying charges. The system quickly settles into a self-consistent state, given by the current profile. As one can easily imagine, it is desirable that all of the plasma be confined to the tokamak. Experiments show that some profiles enhance confinement. Particularly stable are hollow profiles: those where the current density is localized off-axis. There is also experimental evidence that tokamaks with a net charge demonstrate enhanced confinement. The following model is therefore of considerable interest because it can accommodate both a hollow profile and a net charge.

We begin by introducing polar coordinates on a cross section of the tokamak, and a collection of number densities $n_x(r)$ of filaments indexed by the charge and current that each carries. Type-x corresponds to (q_x, j_x). Cylindrical symmetry is assumed throughout. (In the case we will be considering in detail later, each filament carries a positive unit of current j_0, so that $j_0\, n_x(r)$ is the current profile.) From now on it will be safe to view the system as a circular dish in which billions of electrons swarm like bacteria, (as long as one remembers that those electrons are actually rods coming out of the page.) The particles movements are based on the values of the electric and magnetic potentials throughout the dish, while the potential at a point is determined by the positions of all the particles. Because we cannot keep track of what each particle is doing, it is fair to pretend that their movements are random, and that the state they settle into is the one of maximum disorder.

The entropy is approximately given by

$$S = -\int dx \cdot d\vec{r} \cdot n_x(r) \log n_x(r)\,. \tag{1}$$

The electric and magnetic potentials at r due to the filaments at r' satisfy

$$\begin{aligned}\Delta\phi_{r'}(r) &= \int dx \cdot q_x n_x(r')\delta(r-r')\\ \Delta\psi_{r'}(r) &= \int dx \cdot j_x n_x(r')\delta(r-r').\end{aligned} \tag{2}$$

The total potentials

$$\begin{aligned}\phi(r) &= \int d\vec{r}\,'\phi_{r'}(r)\\ \psi(r) &= \int d\vec{r}\,'\psi_{r'}(r)\end{aligned} \tag{3}$$

satisfy Poisson's equation

$$\begin{aligned}\Delta\phi(r) &= \int dx \cdot q_x n_x(r)\\ \Delta\psi(r) &= \int dx \cdot j_x n_x(r).\end{aligned} \tag{4}$$

The electric and magnetic energies of the system are

$$\begin{aligned}E_e &= \int dx \cdot d\vec{r} \cdot q_x n_x(r)\phi(r)\\ E_m &= \int dx \cdot d\vec{r} \cdot j_x n_x(r)\psi(r).\end{aligned} \tag{5}$$

The total charge and current are

$$\begin{aligned}Q &= \int dx \cdot d\vec{r} \cdot q_x n_x(r)\\ I &= \int dx \cdot d\vec{r} \cdot j_x n_x(r).\end{aligned} \tag{6}$$

Our first equation is now obtained by varying $n_x(r)$ and requiring, to first order in $\delta n_x(r)$, the change in S to be a sum of terms each proportional to the change in one of the quantities being held fixed. The result is

$$n_x(r) = \exp\{-\beta_e q_x\phi - \beta_m j_x\psi - \mu_e q_x - \mu_m j_x + 1\}, \tag{7}$$

where β and μ are the standard Lagrange multipliers. β is mathematically identical to a temperature. The electric and magnetic temperatures are allowed to be different. One can imagine characterizing a system by its temperature instead of its energy. However, the correspondence between β and E need be neither one-to-one nor onto. In practice, it is easier to stipulate a temperature, and then solve (4) and (7) for ϕ and ψ, from which all quantities of interest can be found. One may or may not care to find E. On the other hand, once we have solved (4) and (7) for n(r) in terms of μ, we may wish to use (6) to eliminate μ in favor of Q and I.

To get an idea of how this model supports various profiles, suppose for the moment there is no current anywhere, and filaments carry either a positive or negative unit of charge; $x = (\pm q_0, 0)$.

$$\Delta\phi = q_0 e^{-\beta_e q_0\phi - q_0\mu + 1} - q_0 e^{+\beta_e q_0\phi + q_0\mu + 1}$$
$$\psi = 0. \tag{8}$$

Because it is nonlinear, (8) has two classes of solutions. Particularly interesting is the class that corresponds to negative β, for that is when the charged rods form clusters[1]. It makes sense to say that the temperature is negative because the entropy of a state with localized charges is less than the entropy of a state where the charges are spread out evenly, and the first requires more energy than the second (remember that temperature is the derivative of entropy with respect to energy). It is still true that our method describes the state of maximum entropy, not minimum.

Now consider the more interesting case x = ($\pm q_0$, j_0). With the substitution

$$\begin{aligned} &\beta_e q_0\phi + q_0\mu_e \to \phi \\ &\beta_m j_0\psi + j_0\mu_m + 1 \to \psi, \end{aligned} \tag{9}$$

(4) becomes

$$\begin{aligned} \Delta\phi &= -2\beta_e q_0^2 e^{-\psi}\sinh\phi \\ \Delta\psi &= 2\beta_m j_0^2 e^{-\psi}\cosh\phi. \end{aligned} \tag{10}$$

We get the following two relationships as immediate consequences:

$$\begin{aligned} &\left(\frac{\Delta\psi}{\beta_m j_0^2}\right)^2 - \left(\frac{\Delta\phi}{\beta_e q_0^2}\right)^2 = 4e^{-2\psi} \\ &\frac{\Delta\psi}{\beta_m j_0^2}\sinh\phi + \frac{\Delta\phi}{\beta_e q_0^2}\cosh\phi = 0. \end{aligned} \tag{11}$$

A linear approximation to (10) will give us a rough idea of how the profile depends on q_0, j_0, β, Q, and I, but will not include solutions in that interesting class particular to the nonlinear equations. However, by knowing under what combinations of the parameters the approximations made in the linear case fail, we can predict when the nonlinear terms become important, and that will be when the rings form.

We proceed to solve (10) when φ is infinitesimally small and $\psi = \psi_0 + \psi_1$, where ψ_0 is the solution when φ equals zero. If we temporarily restrict ourselves to positive temperatures, $\Delta\psi_0 = \lambda_m e^{-\psi_0}$ implies

$$\psi_0 = 2\log\left(1 + \frac{\lambda_m}{8} r^2\right) \tag{12}$$

($\lambda_m = 2\beta_m j_0^2$ and $\lambda_e = 2\beta_e q_0^2$).
Equations (10) become

$$\begin{aligned} \Delta\phi &= -\lambda_e e^{-\psi_0}\phi \\ \Delta\psi_1 &= \lambda_m e^{-\psi_0}(-\psi_1 + \phi^2). \end{aligned} \tag{13}$$

These may be written more suggestively as

$$\left[\Delta + \frac{\lambda_e}{(1 + \frac{\lambda_m}{8} r^2)^2}\right]\phi = 0 \tag{14}$$

$$\left[\Delta + \frac{\lambda_m}{(1 + \frac{\lambda_m}{8} r^2)^2}\right]\psi_1 = \lambda_m e^{-\psi_0}\phi^2 \equiv f. \tag{15}$$

Let us examine (14) more closely. For small r, $\phi \approx \phi(0) J_0(r/\sqrt{\lambda_e})$, since $\exp(-\psi_0)$ is approximately 1 (J_0 is the zeroth order Bessel function). The second solution is ruled out because the derivative of φ, the electric field, must be zero at the origin. For medium r, the WKB method of solving differential equations is appropriate, as $\exp(-\psi_0)$ acts like a small potential. In the limit of small r, the coefficient of the leading r term in the WKB solution must match the coefficient of the leading r term in the J_0 solution. One finds

$$\phi = \frac{-\phi(0)}{2\lambda_e(\frac{\lambda_m}{8} + \lambda_e)}\sqrt{1 + \frac{\lambda_m}{8} r^2}\,\cos\left(\sqrt{\frac{8\lambda_e}{\lambda_m}}\tan^{-1}\left(\sqrt{\frac{\lambda_m}{8}} r\right)\right). \tag{16}$$

For large r, $\Delta\phi \approx 0$ implies

$$\phi = c_0 \log r + \sum_{n=1} \frac{c_n}{r^n}. \tag{17}$$

The log term will be present only when the net charge Q is nonzero. The power terms are like multipole corrections. To find the c's, one needs to adjust φ(0) in order that Δφ satisfy (6). This is best done numerically.

To find ψ_1 from (15), we need the Green's function

$$\left[\Delta + \frac{\lambda_m}{(1 + \frac{\lambda_m}{8} r^2)^2}\right] G(r, r') = \delta(r - r'). \tag{18}$$

In the two regions to the left and right of r', G is some linear combination of the two homogenous solutions

$$\begin{aligned} &\left[\Delta + \frac{\lambda_m}{(1 + \frac{\lambda_m}{8} r^2)^2}\right] g(r) = 0. \\ &G = \begin{cases} a_1 g_1 + a_2 g_2, & r - r' < 0 \\ b_1 g_1 + b_2 g_2, & r - r' > 0. \end{cases} \end{aligned} \tag{19}$$

If we ask that G be continuous we obtain a relation between the a's and b's. A second relation comes from integrating (18) from r'-ε to r'+ ε:

$$\Delta(r\partial_r G) = 1, \tag{20}$$

where here Δ represents the jump that occurs at r' (any continuous function contributes nothing). Finally, ψ_1 is given by

$$\psi_1(r) = \int dr' G(r,r') f(r'). \qquad (21)$$

This is a work in progress. The very next thing to do would be to compute ψ_1 assuming all terms of the form $(1+\frac{\lambda_m}{8}r^2)^{-2}$ are constants. Then, after the solution is found, to see what the effect of changing those constants would be. Another easy thing to do would be to solve for ψ_1 when β_e is negative.

If there are no charges, the trivial solution to (15) is $\psi_1 = 0$, as it must be. The introduction of an electric potential causes the current-carrying filaments to adjust themselves slightly, but only because they have a charge "tied on". If there were two species of objects, those that carry current and those that carry charge, then no coupling would take place. One can imagine charges jumping from one filament to another, without affecting the current. Such phenomena are not included in the model presented here. Still, much can be done with what we have. To deal with variable strength filaments simply let x = (mq_0, nj_0), m,n e {±1, ±2, …}. More importantly, the parameter domain remains to be explored. For instance, it would be nice to see what the charge and current profiles look like at different values of β_e/β_m and q_0/j_0, especially near when our approximations fail. Hopefully future efforts to control the current profile through an understanding of the solution to the nonlinear equations (10) will benefit from the above analysis.

*Research supported in part by the National Science Foundation under the REU Physics program. This work supervised by Professor G. Morales.

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