Simulation of Shear Alfvén Waves in LAPD using the BOUT++ code

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Abstract

The linear and nonlinear physics of shear Alfvén waves are investigated using the 3D Braginskii fluid code BOUT++. The code has been verified against analytical calculations for the dispersion of kinetic and inertial Alfvén waves. Various mechanisms for forcing Alfvén waves in the code are explored, including introducing localized current sources similar to physical antennas used in experiments. Using this foundation, the code is used to model nonlinear interactions among shear Alfvén waves in a cylindrical magnetized plasma, such as that found in the Large Plasma Device (LAPD) at UCLA. In the future this investigation will allow for examination of the nonlinear interactions between shear Alfvén waves in both laboratory and space plasmas in order to compare to predictions of MHD turbulence.

1 Introduction

The study of shear Alfvén waves has been an ongoing scientific endeavor since 1942 when Hannes Alfvén described them as a type of MHD wave [1], and Lundquist produced them in a laboratory setting in 1949 [2]. They have long been theorized to be the source of a number of interstellar and solar phenomena such as the Earth's aurora [8] and have only very recently been observationally proven to be the source of solar winds emanating from the solar corona [3]. They are also prevalent in laboratory plasmas [4] and can provide valuable insight into containing plasmas for fusion-related applications.

Experimentally, the nonlinear interactions of Alfvén waves have been studied, specifically at the Large Plasma Device (LAPD) at UCLA [5]. The results of the Carter group's investigation into the pseudomode at beat frequency generated by two Alfvén waves is the motivation for this computational study of how Alfvén waves interact in a BOUT++ simulation. The BOUT++ codes is a Braginskii two-fluid model [6], and was developed by Ben Dudson of the University of York. It is the successor to the BOUT code, which has been used in the study of plasma instabilities for the Large Plasma Device [7].

In this paper, a computational simulation of the linear propagation of shear Alfvén waves with the BOUT++ code is presented, as well as a brief introduction to the preliminary nonlinear interactions that have been produced in these simulations. While there have been previous computational 3-dimensioinal studies [9] into the generation of shear Alfvén waves, this is the first to be done with the BOUT++ code in an LAPD plasma. In Section 2, a linear verification of the ability of BOUT++ to match an analytic dispersion relation is presented. Section 3 describes how a linear forcing term, simulating an experimental antenna, was used in the BOUT++ code to generate and drive a continuous Alfvén wave. The resonant response of the Alfvén wave is examined both analytically and with the code, giving another linear verification of the capabilities of BOUT++. Finally, an initial nonlinear simulation is examined in Section 4, building on the foundation of linear verification models that precede it.

The main focus of this paper is to gain an understanding of the ability of BOUT++ to simulate Alfvén Waves linearly so that there is a validated framework to undergo studies of non-linear interactions.

2 Linear Dispersion Relation

BOUT++ is an initial value solver which is able to solve the coupled non-linear partial differential electromagnetic and fluid differential equations that describe the dynamics of Alfvén waves in a plasma.

The first of these equations describes the continuity of particles in a plasma:

Density Equation:

$$\frac{\partial N_i}{\partial t} = \nabla_{\parallel}(j_{\parallel}/e) \tag{1}$$

Next is a parallel force equation for electrons:

Electron Parallel Momentum:

$$m_e \frac{\partial V_{\parallel e}}{\partial t} = e \left(\partial_{\parallel} \phi + \frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} \right) \quad (2)$$

Then there is the vorticity, or the curl of the velocity of the fluid. This quantity can also be thought of as the angular rate of rotation in a fluid:

Potential Vorticity

$$\frac{\partial \varpi}{\partial t} = N_{i0} Z_i e \frac{4\pi V_A^2}{c} \nabla_{\parallel} j_{\parallel} \quad (3)$$

Finally, there is Ampere's Law, which is solved by inverting the Laplacian in BOUT++:

Ampere's Law

$$\nabla_{\perp}^2 A_{\parallel} = \frac{-4\pi}{c} j_{\parallel} \quad (4)$$

The following two definitions are also used in BOUT++:

Potential Vorticity Definition:

$$N_{i0} Z_{ie} \nabla_{\perp}^2 \phi = \varpi \quad (5)$$

Current Density:

$$j_{\parallel} = -e N_{i0} V_{\parallel e} \quad (6)$$

Combining these linear equations using a Fourier Transform, The following dispersion relation for Alfvén waves is obtained:

$$\frac{\omega^2}{k_{\parallel}^2} = \frac{V_A^2 + V_{te}^2 \left(\frac{k_{\perp}^2 c^2}{\omega_p e^2} \right)}{1 + \left(\frac{k_{\perp}^2 c^2}{\omega_p e^2} \right)} \quad (7)$$

This analytical solution was plotted, relating the angular frequency ω of an Alfvén wave to its wave number $k_{\perp}$ at different electron thermal speeds (T_e = 0, 10, and 100 eV) . A spatial wave structure with specific parallel and perpendicular wavelength is initialized in the simulation and allowed to propagate. The frequency of propagation is observed and plotted against the perpendicular wave number in Figure 1 alongside the analytic dispersion relation.

3 Sine Forcing Term

3.1 1-D Sine forcing term

Next, instead of simply placing an Alfvén wave in the simulation with initial conditions, a continuous driving force is simulated by putting an antenna into the code. The objective is to verify the BOUT++ code against analytic calculation. In order to examine this more realistic model, assumptions are made that there is no evolving density ($\frac{\partial N_i}{\partial t}$) term and that m_e is zero. A sin forcing term is added onto $A_{j\parallel}$, and the wave is driven off-resonance in order to prevent exponential growth. The following normalized (from equations 1 - 4) were solved in BOUT++:

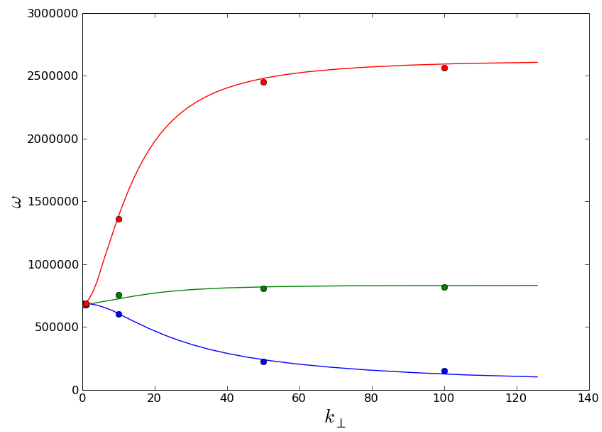

Figure 1: ω vs. $k_{\perp}$ comparision of analytic calculations (curves) (Blue - T_e = 0, Green - T_e = 10, and Red - T_e = 100 eV) and by BOUT++ (points).

$$\frac{\partial A_{j\parallel}}{\partial t} = -\mu \nabla_{\parallel} \phi + C \sin(\omega t) \sin(kx) \quad (8)$$

$$\frac{\partial \varpi}{\partial t} = \nabla_{\parallel} j_{\parallel} \quad (9)$$

$$\nabla_{\perp}^2 A_{\parallel} = \frac{-\beta\mu}{2} j_{\parallel} \quad (10)$$

$$\nabla_{\perp}^2 \phi = \varpi \quad (11)$$

The time derivative of equation 8 is taken, and equations 9, 10 and 11 are used along with the following relation:

$$A_{j\parallel} = -j_{\parallel} - A_{\parallel} \quad (12)$$

This results in the following partial differential equation, Equation 13 :

$$\frac{2}{\beta\mu} \frac{\partial^2 \nabla_{\perp}^2 A_{\parallel}}{\partial t^2} - \frac{\partial^2 A_{\parallel}}{\partial t^2} + \frac{2}{\beta} \nabla_{\parallel}^2 A_{\parallel} \quad (13)$$

$$-\omega C \cos(wt)\sin(kx) = 0$$

Equation 13 is solved by doing a Fourier Transform in all three variables:

$$A_\parallel : A_\parallel(x, y, t) \longrightarrow \hat{A}_\parallel(\varphi, \xi, \tau)$$

Using an Inverse Fourier Transform, $A_\parallel$ is obtained from $\hat{A}_\parallel$:

$$A_\parallel = \frac{\omega C}{2} \frac{\sin(kx + \omega t) + \sin(kx - \omega t)}{\left(\omega^2 - \frac{2}{\beta}k^2\right)} \tag{14}$$

Plotting this analytic solution with Python (blue) on the same plot as the solution given by BOUT++ (green), Figure 2 is obtained:

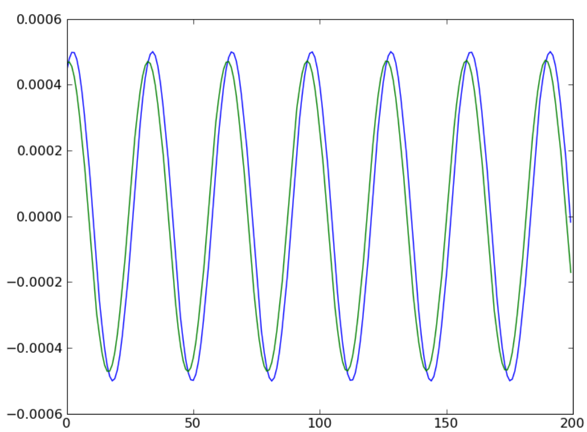

Figure 2: Plot comparing analytic and computational results of $A_\parallel$ when a forcing sin term and a damping term are added in. The BOUT++ results are in blue..

3.2 2-D Sine forcing term

Next, the extra term, $\sin(k_\perp y)$, is added to the forcing term in order to make the antenna a more realistic model which creates Alfvén Waves in a 2-D plane. A damping term is added in order to prevent exponential growth, and this time the wave is driven on-resonance:

$$\frac{\partial A_\parallel}{\partial t} = -\mu \nabla_\parallel \phi + C\sin(\omega t)\sin(kx)\sin(k_\perp y) + 0.51\nu_{ei} j_\parallel \tag{15}$$

The Non-homogeneous linear partial differential equation, Equation 16, is determined as before in Section 3.1:

$$\frac{2}{\beta\mu}\frac{\partial^2 \nabla_\perp^2 A_\parallel}{\partial t^2} - \frac{\partial^2 A_\parallel}{\partial t^2} + \frac{2}{\beta}\nabla_\parallel^2 A_\parallel + \frac{\nu_{ei}}{\beta\mu}\frac{\partial \nabla_\perp^2 A_\parallel}{\partial t} \tag{16}$$
$$-\omega C\cos(wt)\sin(kx)\sin(k_\perp y) = 0$$

Using the same Fourier Transform Method as used in Section 3.1, the following answer is obtained:

$$A_\parallel = -\omega C \left[\frac{\sin(kx)\sin(k_\perp y)\left((\varphi)\cos(\omega t) + \left(\frac{\nu_{ei}}{\beta\mu}\omega k_\perp^2\right) sin(\omega t)\right)}{(\varphi)^2 + \left(\frac{\nu_{ei}}{\beta\mu}\omega k_\perp^2\right)^2}\right] \tag{17}$$

$$\text{where } \varphi = \frac{2}{\beta\mu}\omega^2 k_\perp^2 + \omega^2 - \frac{2}{\beta}k^2 \tag{18}$$

The steady state condition found in BOUT++ is plotted over this analytical answer. There is good agreement between the two, as seen in Figure 3.

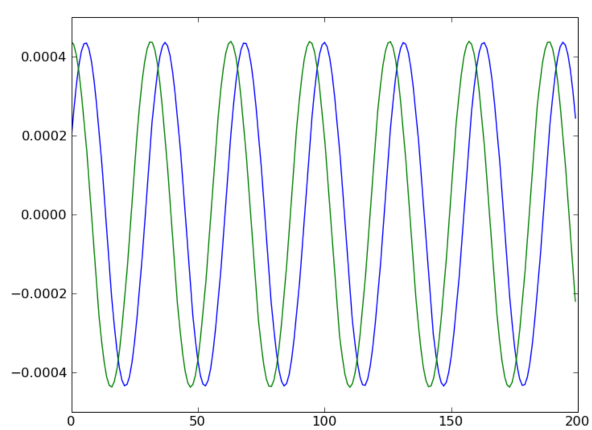

Figure 3: plot comparing Analytic and computational results of $A_\parallel$ when a forcing sin forcing term (in both the x and y direction) and a damping term are added in. The BOUT++ results are in blue.

Experimentally, this simulation was meant to model driving a current pulse in the magnetic field and sending Alfvén waves through the plasma. The driving sine term was put in the equation for the Vector Potential since varying this parameter most directly resembled a perturbation to the magnetic field. There is a shift in phase due to the fact that the BOUT++ simulation started at a different point, although this has no physical or computational bearing on the results of this study.

3.3 Resonance Frequency

Plotting an analytic spectrum of $A_{\parallel}$ in relation to the angular frequency, there is determined to be a value of ω at which the Alfvén wave is driven at resonance. A series of discrete frequencies were used as the driving frequency for generating an Alfvén wave in BOUT++ and the resulting $A_{\parallel}$ amplitudes were plotted in the same figure, Figure 4.

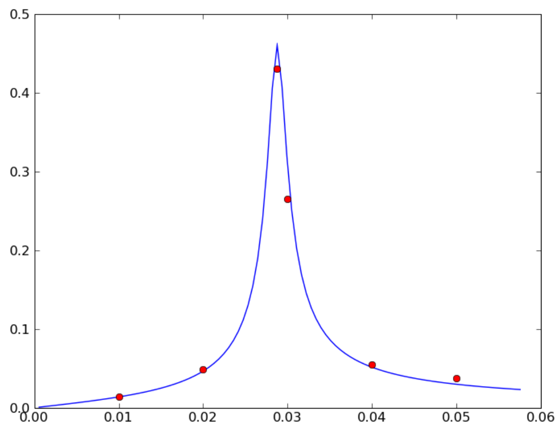

Figure 4: Plot of Apar vs. omega, which shows a resonance frequency, which is reached when omega satisfies the dispersion relation, given a fixed $k_{\parallel}$.

The BOUT++ results are again verified by analytic calculation, and the peak amplitude for $A_{\parallel}$ is reached when ω is a value such that the term Equation 18 is zero. This occurs when the ω satisfies the dispersion relation, Equation 7, demonstrating how the natural response of a plasma under perturbation is to generate an Alfvén wave.

4 Non-Linear Simulation

After successfully demonstrating the capabilities of BOUT++ to simulate a continuous driving force that creates Alfvén waves, the next step was to simulate the non-linear reactions that underlie Alfvénic turbulence.

The aim of this simulation, modeled on the experimental precedence produced by the Carter group [5], was to generate non-driven frequencies, which would indicate non-linear interactions and coupling by driven waves. Following is the analytic prediction and of what is expected from the simulations of BOUT++.

At first, only one wave is theoretically driven and when the advection term is turned on in the vorticity equation, Equation 19 is obtained.

$$\frac{\partial \varpi}{\partial t} = V_E \cdot \nabla \varpi \tag{19}$$

This non-linear addition, along with the definition of electron velocity (Equation 20) gives a final non-linear differential equation with 2 fluctuating quantities, ϕ and ϖ, multiplied by each other (Equation 21)

$$V_E \cdot \nabla \varpi = (\overrightarrow{B} \times \nabla \phi) \cdot \nabla \varpi \tag{20}$$

$$\frac{\partial \varpi}{\partial t} = \left(\frac{\partial \phi}{\partial x} \frac{\partial \varpi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \varpi}{\partial x} \right) \tag{21}$$

Using Fourier transforms to look at the individual wave interactions, The Equations 22 and 23 are used to change Equation 21 into the form of Equation 24.

Fourier Transforms:

$$\phi = \sum_{k,\omega} \tilde{\phi}_{k,\omega} e^{i(kr-\omega t)} \tag{22}$$

$$\varpi = \sum_{k\prime,\omega\prime} \tilde{\varpi}_{k,\omega} e^{i(k\prime r-\omega\prime t)} \tag{23}$$

The first term of this Fourier transformed equation is:

$$i\omega \tilde{\varpi}_{k,\omega} = \sum_{k\prime,\omega\prime} k_{x\prime}(k_y - k_{y\prime}) \tilde{\phi}_{k\prime,\omega\prime} \tilde{\varpi}_{k-k\prime,\omega-\omega\prime} \tag{24}$$

The most revealing result is an intermediate expression (Equation 25) in which there are δ-functions indicating that waves can only exist when k, $k\prime$, and $k\prime\prime$ as well as ω $\omega\prime$, and $\omega\prime\prime$ satisfy the three-wave coupling condition.

$$\sum_{k,k\prime,\omega,\omega\prime} k_x k_y \tilde{\phi}_{k,\omega} \tilde{\varpi}_{k\prime,\omega\prime} \delta(k + k\prime - k\prime\prime) \delta(\omega + \omega\prime - \omega\prime\prime) \tag{25}$$

When this non-linear advection term is turned on in BOUT++ and the same continuous driving force is simulated as described in Section 3, a Power Spectrum for $A_{\parallel}$ is obtained for different values of angular frequency, ω, which indicates non-linear effects taking place (Figure 5).

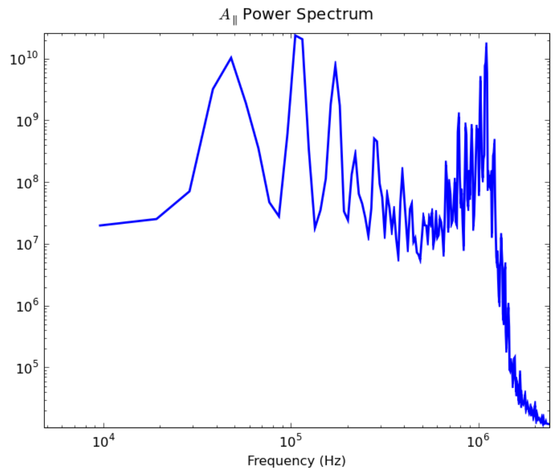

Figure 5: Non - Linear Spectrum.

When compared to the linear Power Spectrum without the non-linear advection term turned on (Figure 6), there are a number of non-driving frequencies ω produced. This indicates that BOUT++ is additionally able to model non-linear wave interactions that could potentially be compared against experimental results.

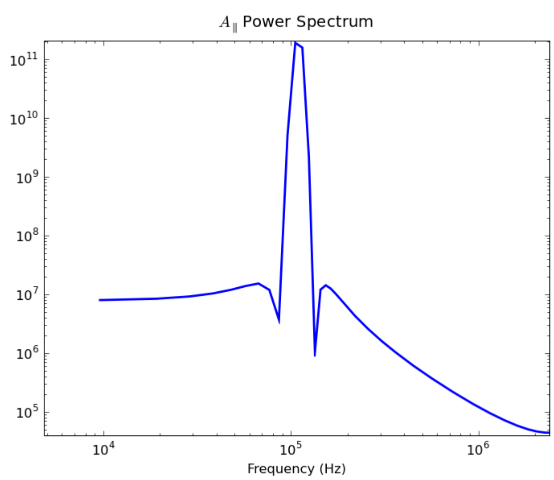

Figure 6: Linear Spectrum.

5 Conclusion

The results of this computational study of the plasma simulation code BOUT++ has verified that it is able to correctly model the dispersion relation for Alfvén waves in a plasma. Additionally it is able to simulate launching a continuous Alfvén wave when a driving force is added to the set of initial equations solved by the code. Eventually, this will evolve into a more realistic model of a physical antenna in experimental studies of Alfvén waves. Finally, it has been determined that BOUT++ is able to simulate non-linear effects of Alfvén waves in a plasma when the non-linear terms of the initial equation set is turned on. Future work revolves around this non-linear work, specifically looking at changing the initial conditions and parameters of these driven waves, and launching multiple waves in order to analyze their non-linear interactions.

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